

Note

The independence number in graphs of maximum degree three

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Abstract

We prove that a K_4 -free graph G of order n , size m and maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n - m - \lambda - tr)$, where λ counts the number of components of G whose blocks are each either isomorphic to one of four specific graphs or edges between two of these four specific graphs and tr is the maximum number of vertex-disjoint triangles in G . Our result generalizes a bound due to Heckman and Thomas [C.C. Heckman, R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, *Discrete Math.* 233 (2001) 233–237].

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We consider finite simple and undirected graphs $G = (V, E)$ of order $n(G) = |V|$ and size $m(G) = |E|$. The independence number $\alpha(G)$ of G is defined as the maximum cardinality of a set of pairwise non-adjacent vertices which is called an independent set.

Our aim in the present note is to extend a result of Heckman and Thomas [6] (cf. Theorem 1) about the independence number of triangle-free graphs of maximum degree at most three to the case of graphs which may contain triangles. With their very insightful and elegant proof, Heckman and Thomas also provide a short proof for the result conjectured by Albertson, Bollobás and Tucker [1] and originally proved by Staton [9] that every triangle-free graph G of maximum degree at most three has an independent set of cardinality at least $\frac{5}{14}n(G)$ (cf. also [7]). (Note that there are exactly two connected graphs for which this bound is best-possible [2,3,5,8] and that Fraughnaugh and Locke [4] proved that every cubic triangle-free graph G has an independent set of cardinality at least $\frac{11}{30}n(G) - \frac{2}{15}$, which implies that, asymptotically, $\frac{5}{14}$ is not the correct fraction.)

In order to formulate the result of Heckman and Thomas and our extension of it we need some definitions.

A block of a graph is called *difficult* if it is isomorphic to one of the four graphs K_3 , C_5 , K_4^* or C_5^* in Fig. 1, i.e., it is either a triangle, or a cycle of length five, or arises by subdividing two independent edges in a K_4 twice, or arises by adding a vertex to a C_5 and joining it to three consecutive vertices of the C_5 . A connected graph is called *bad* if its blocks are either difficult or are edges between difficult blocks.

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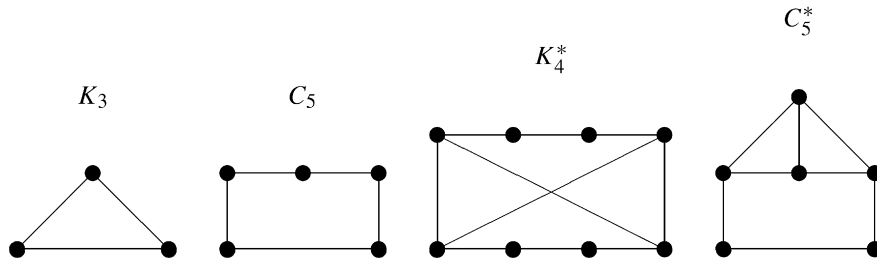


Fig. 1. Difficult blocks.

For a graph G we denote by $\lambda(G)$ the number of components of G which are bad and by $\text{tr}(G)$ the maximum number of vertex-disjoint triangles in G . Note that for triangle-free graphs G our definition of $\lambda(G)$ coincides with the one given by Heckman and Thomas [6]. Furthermore, note that $\text{tr}(G)$ can be computed efficiently for a graph G of maximum degree at most three as it equals exactly the number of non-trivial components of the graph formed by the edges of G which lie in a triangle of G .

Theorem 1 (Heckman and Thomas [6]). *Every triangle-free graph G of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G))$.*

Since every K_4 in a graph of maximum degree at most three must form a component and contributes exactly one to the independence number of the graph, we can restrict our attention to graphs that do not contain K_4 's.

Theorem 2. *Every K_4 -free graph G of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G) - \text{tr}(G))$.*

Proof. For a graph G we denote the quantity $4n(G) - m(G) - \lambda(G) - \text{tr}(G)$ by $\psi(G)$. We wish to show that $7\alpha(G) \geq \psi(G)$. For contradiction, we assume that $G = (V, E)$ is a counterexample to the statement such that $\text{tr}(G)$ is smallest possible and subject to this condition the order $n(G)$ of G is smallest possible. If $\text{tr}(G) = 0$, then the result follows immediately from Theorem 1. Therefore, we may assume $\text{tr}(G) \geq 1$. Since $\alpha(G)$ and $\psi(G)$ are additive with respect to the components of G , we may assume that G is connected. Furthermore, we may clearly assume that $n(G) \geq 4$.

Claim 1. *Every vertex in a triangle has degree three.*

Proof of Claim 1. Let x, y and z be the vertices of a triangle. We assume that $d_G(x) = 2$. Clearly, the graph $G' = G[V \setminus \{x, y, z\}]$ is no counterexample, i.e., $7\alpha(G') \geq \psi(G')$. Since for every independent set I' of G' , the set $I' \cup \{x\}$ is an independent set of G , we have $\alpha(G) \geq \alpha(G') + 1$. The triangle xyz is vertex-disjoint from all triangles in G' , and so $\text{tr}(G) \geq \text{tr}(G') + 1$.

Suppose $\min\{d_G(y), d_G(z)\} = 2$. Then $\max\{d_G(y), d_G(z)\} = 3$, since G is not just a triangle. Furthermore, by the definition of a bad graph, we have $\lambda(G') = \lambda(G)$ and obtain

$$\begin{aligned} 7\alpha(G) &\geq 7\alpha(G') + 7 \\ &\geq \psi(G') + 7 \\ &= 4n(G') - m(G') - \lambda(G') - \text{tr}(G') + 7 \\ &\geq 4(n(G) - 3) - (m(G) - 4) - \lambda(G) - (\text{tr}(G) - 1) + 7 \\ &\geq \psi(G) - 12 + 4 + 1 + 7 \\ &= \psi(G), \end{aligned}$$

which implies a contradiction. Therefore, we may assume $d_G(y) = d_G(z) = 3$. Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Regardless of whether $y' = z'$ or not, we have $\text{tr}(G) \geq \text{tr}(G') + 1$.

If $y' = z'$, then G' is connected, y' is a vertex of degree one in G' and thus $\lambda(G') = \lambda(G) = 0$. If $y' \neq z'$ and $\lambda(G') \geq 2$, then $\lambda(G') = 2$ and G is a bad graph itself, i.e., $\lambda(G) = 1$. Therefore, in both cases $\lambda(G') \leq \lambda(G) + 1$,

and we obtain

$$\begin{aligned}
 7\alpha(G) &\geq 7\alpha(G') + 7 \\
 &\geq \psi(G') + 7 \\
 &= 4n(G') - m(G') - \lambda(G') - \text{tr}(G') + 7 \\
 &\geq 4(n(G) - 3) - (m(G) - 5) - (\lambda(G) + 1) - (\text{tr}(G) - 1) + 7 \\
 &\geq \psi(G) - 12 + 5 - 1 + 1 + 7 \\
 &= \psi(G),
 \end{aligned}$$

which implies a contradiction and the proof of the claim is complete. \square

Claim 2. No two triangles of G share an edge, i.e., G does not contain $K_4 - e$.

Proof of Claim 2. Let x, y, y' and z be such that xyy' and $yy'z$ are triangles. Let $G' = G[V \setminus \{y'\}]$. Clearly, $\alpha(G) \geq \alpha(G')$, $\text{tr}(G) \geq \text{tr}(G') + 1$ and G' is connected. Note that, by Claim 1, both x and z have degree 3 in G and thus x, y and z are all of degree 2 in G' .

If G' is bad, then x, y and z are three consecutive vertices in a block of G' isomorphic to C_5 . Since the corresponding block in G is isomorphic to C_5^* , the graph G is also bad. Conversely, if G is bad, then x, y, y' and z belong to a block of G isomorphic to C_5^* . Since the corresponding block in G' is isomorphic to C_5 , the graph G' is also bad.

Therefore, $\lambda(G') = \lambda(G)$, and we obtain

$$\begin{aligned}
 7\alpha(G) &\geq 7\alpha(G') \\
 &\geq \psi(G') \\
 &= 4n(G') - m(G') - \lambda(G') - \text{tr}(G') \\
 &\geq 4(n(G) - 1) - (m(G) - 3) - \lambda(G) - (\text{tr}(G) - 1) \\
 &\geq \psi(G) - 4 + 3 + 1 \\
 &= \psi(G),
 \end{aligned}$$

which implies a contradiction, and the proof of the claim is complete. \square

Note that, by Claim 2, adding an edge to a subgraph of G cannot create a K_4 .

Let xyz be a triangle in G . By Claim 1, we have $N_G(x) = \{x', y, z\}$, $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$ and, by Claim 2, x', y' and z' are all distinct. Let $G' = G[V \setminus \{x, y, z\}]$.

Claim 3. The set $\{x', y', z'\}$ is independent.

Proof of Claim 3. For contradiction, we assume that $x'y' \in E$. For every independent set I' of G' either $I' \cup \{x\}$ or $I' \cup \{y\}$ is an independent set of G , which implies $\alpha(G) \geq \alpha(G') + 1$. Since G' has at most two components, we have $\lambda(G') \leq \lambda(G) + 2$. Furthermore, $n(G') = n(G) - 3$, $m(G') = m(G) - 6$, $\text{tr}(G) \geq \text{tr}(G') + 1$, and we obtain a similar contradiction as before, which completes the proof of the claim. \square

Claim 4. There are two edges e and f in $\{x'y', y'z', x'z'\}$ such that $\lambda(G' + e) \leq \lambda(G) + 1$ and $\lambda(G' + f) \leq \lambda(G) + 1$.

Proof of Claim 4. For contradiction, we assume that $\lambda(G' + x'y') \geq \lambda(G) + 2$. This implies that G' consists exactly of two bad components and that G itself is not a bad graph. Hence $x'y'$ cannot be an edge between two difficult blocks, since otherwise G would be a bad graph. Thus both $G' + x'z'$ and $G' + y'z'$ are connected, and the claim follows for $\{e, f\} = \{x'z', y'z'\}$. \square

Claim 5. If $\lambda(G' + e) = \lambda(G' + f) = \lambda(G) + 1$, then either $\text{tr}(G' + e) \leq \text{tr}(G) - 1$ or $\text{tr}(G' + f) \leq \text{tr}(G) - 1$.

Proof of Claim 5. We may assume that $e = x'z'$ and $f = y'z'$. For contradiction, we assume that $\text{tr}(G' + e), \text{tr}(G' + f) \geq \text{tr}(G)$. This implies that x' and z' have a common neighbour x'' in G' and that y' and z' have a common neighbour y'' in G' . If possible, we choose $x'' = y''$. Clearly, this implies that G' is connected. Furthermore, since the vertices $x, y, z, x', y', z', x'', y''$ all lie in one block of G which cannot be a bad block, the graph G cannot be a bad graph. Since $\lambda(G' + e) = \lambda(G' + f) = \lambda(G) + 1$, both $G' + e$ and $G' + f$ must be bad graphs.

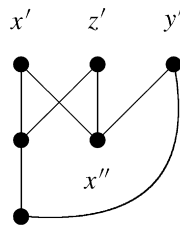


Fig. 2.

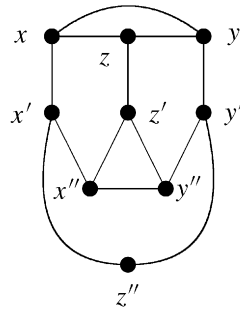


Fig. 3.

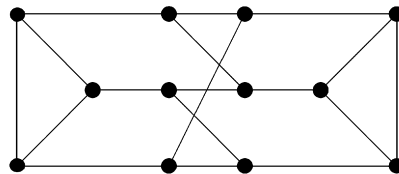


Fig. 4.

If the triangle $x'z'x''$ forms a difficult block in $G' + e$, the edge $x'x''$ forms a block in $G' + f$ which does not connect two difficult blocks. This implies that $G' + f$ cannot be bad, which is a contradiction. Therefore, by symmetry, we may assume that the triangle $x'z'x''$ is contained in a difficult block B_e in $G' + e$ which is isomorphic to C_5^* and that also the triangle $y'z'y''$ is contained in a difficult block B_f in $G' + f$ which is isomorphic to C_5^* . First, we assume $x'' = y''$. If $e = x'z'$ is not the edge shared by the two triangles of B_e , then either x' and x'' or z' and x'' have a common neighbour in G' . This implies that y' is adjacent to either x' or z' , which contradicts Claim 3. Hence the edge $e = x'z'$ must be the edge shared by the two triangles of B_e . Now, G' contains the configuration shown in Fig. 2. Clearly, all six vertices in Fig. 2 belong to one block of $G' + f$, which cannot be a difficult block. Therefore, $G' + f$ cannot be a bad graph, which is a contradiction.

Next, we assume that $x'' \neq y''$. By the choice of x'' and y'' , this implies that no vertex in G' is adjacent to all of x', y' and z' . If $e = x'z'$ is the edge shared by the two triangles of B_e , then x' and z' must have a common neighbour in G' different from x'' . This implies that y'' is adjacent to all of x', y' and z' , which is a contradiction. Hence $x'z'$ is not the edge shared by the two triangles of B_e . If $x'x''$ is the edge shared by the two triangles of B_e , then the block of $G' + f$ which contains x' contains two vertex-disjoint triangles. Therefore, $G' + f$ cannot be a bad graph, which is a contradiction. We obtain that $z'x''$ is the edge shared by the two triangles of B_e , which implies the existence of a vertex z'' such that G contains the configuration shown in Fig. 3.

Since $G[\{x, y, z, x', y', z', x'', y'', z''\}]$ is not a counterexample, the vertex z'' has degree three. Now the graph $G'' = G[V \setminus \{x, y, z, x', y', z', x'', y'', z''\}]$ satisfies $\alpha(G) \geq \alpha(G'') + 3$, $n(G) = n(G'') + 9$, $m(G) = m(G'') + 14$, $\lambda(G'') \leq \lambda(G) + 1$ and $\text{tr}(G) \geq \text{tr}(G'') + 2$, which implies a similar contradiction as before and completes the proof of the claim. \square

Note that $\text{tr}(G' + x'z') \leq \text{tr}(G') + 1 = \text{tr}(G)$. Therefore, by Claims 4 and 5, we can assume that either $\lambda(G' + x'z') \leq \lambda(G)$ and $\text{tr}(G' + x'z') \leq \text{tr}(G)$ or $\lambda(G' + x'z') = \lambda(G) + 1$ and $\text{tr}(G' + x'z') \leq \text{tr}(G) - 1$, both of which

imply that $\lambda(G' + x'z') + \text{tr}(G' + x'z') \leq \lambda(G) + \text{tr}(G)$. Similarly as above, for every independent set I' of $G' + x'z'$ either $I' \cup \{x\}$ or $I' \cup \{z\}$ is an independent set of G , which implies $\alpha(G) \geq \alpha(G') + 1$. Since $n(G' + e) = n(G) - 3$ and $m(G' + e) = m(G) - 5$, we obtain a similar contradiction as above, which completes the proof. \square

Note that [Theorem 2](#) is best-possible for all bad graphs, all graphs which arise by adding an edge to a bad graph and further graphs such as for instance the graph in [Fig. 4](#).

In [\[5\]](#) Heckman characterized the extremal graphs for [Theorem 1](#). Similarly, it might be an interesting yet challenging task to characterize the extremal graphs for [Theorem 2](#).

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